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L^1 DECAY PROPERTIES FOR A SEMILINEAR PARABOLIC SYSTEM

BY

L. AMOUR AND T. RAOUX

Laboratoire de Mathématiques, CNRS - UPRESA 6056, Université de Reims Moulin de la Housse, B.P. 1039, 51687 Reims Cedex 2, France
e-mail: laurent.amour@univ-reims.fr, thierry.raoux@univ-reims.fr

ABSTRACT

This article is concerned with the decay property in the L^1 norm as $t \to \infty$ of the nonnegative solutions of the initial value problem in $\mathbb{R}^n$

$$\begin{cases} u_t = \Delta u + \mu|\nabla v|^q \\ v_t = \Delta v + \nu|\nabla u|^p \end{cases}$$

for different values of the parameters $p, q \geq 1$ and when $\mu, \nu < 0$. If

$$pq > \frac{\inf(p,q)}{n+1} + (n+2)/(n+1)$$

then $\lim_{t\to+\infty} \|u(t) + v(t)\|_1 > 0$ and when

$$pq < \frac{\inf(p,q)}{n+1} + (n+2)/(n+1)$$

then $\lim_{t\to+\infty} \|u(t) + v(t)\|_1 = 0$.

1. Introduction and results

We are interested in the coupled nonlinear parabolic equations

$$\begin{cases} u_t = \Delta u + \mu|\nabla v|^q \\ v_t = \Delta v + \nu|\nabla u|^p \end{cases} \tag{1}$$

with $t \geq 0$, $x \in \mathbb{R}^n$, $p, q \geq 1$ and $\mu, \nu < 0$.

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Our main result determines L^1 decay properties of nonnegative solutions to the coupled system (1) for different values of the parameters p and q. It is an extension of some recent results concerning the nonnegative solutions of the scalar equation

$$(2) \qquad u_t = \Delta u + \mu |\nabla u|^q,$$

where $q \geq 1$ and $\mu < 0$.

Existence and L^1 decay properties for the solutions of (2) are solved in [AB] and [BK] for regular initial data. There exist unique global solutions to the Cauchy problem with initial data in $C_b^2(\mathbb{R}^n) = C^2(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ for any $\mu \neq 0$ and any $q \geq 1$. Moreover, for $\mu < 0$ and for nonnegative initial data in $C_b^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, it is proved that $\lim_{t \to +\infty} \|u(t)\|_{L^1(\mathbb{R}^n)} = 0$ when $1 \leq q \leq (n+2)/(n+1)$ and $\inf_{t \geq 0} \|u(t)\|_{L^1(\mathbb{R}^n)} > 0$ when $q > (n+2)/(n+1)$. These results are extended to irregular initial data in [BL] and [BSW]. For $q = 1$, more general results and more precise estimates are obtained in [BGL] and [BRV]. In the case $\mu > 0$, large time behaviour of the solutions is given in [GGK]. Existence and L^1 decay for $u_t = \Delta u - a(x) u^p |\nabla u|^q$ are studied in [P]. See also [A] in the case of bounded domains of $\mathbb{R}^n$.

Let us also mention that nonlinear parabolic systems without gradient terms, namely $u_t = \Delta u + u^{p_{11}} v^{p_{12}}$, $v_t = \Delta v + u^{p_{21}} v^{p_{22}}$, have been largely studied. See, e.g., [EH], [EL] and [L] and references cited there.

When one passes from equation (2) to system (1), we lose the maximum and minimum principle properties. However, some elementary results concerning the nonnegative solutions are preserved:

(i) The solutions to (1) satisfy $u(x,t) \leq (e^{t\Delta} u_0)(x)$, $v(x,t) \leq (e^{t\Delta} v_0)(x)$ as long as they exist.

(ii) The quantities $\|u(t)\|_{L^1(\mathbb{R}^n)}$ and $\|v(t)\|_{L^1(\mathbb{R}^n)}$ are nonincreasing functions of $t \geq 0$.

(iii) The following integrability property holds,

$$(3) \qquad \int_0^{+\infty} \int_{\mathbb{R}^n} (|\nabla u|^p + |\nabla v|^q) \, dx \, dt < \infty.$$

By a proper scaling, namely $\tilde{u} = \alpha u(\varepsilon x, \varepsilon^2 t)$, $\tilde{v} = \beta v(\varepsilon x, \varepsilon^2 t)$, we suppose $\mu = \nu = -1$. Then, taking account of the symmetry in (1) we suppose without loss of generality that $p \leq q$.

Let $\mathcal{C}^+$ be the unbounded subdomain of $\mathcal{U} = \{(p,q),\ p \geq 1,\ q \geq 1,\ p \leq q\}$

defined by

$$\mathcal{C}^+ = \Big\{(p,q) \in \mathcal{U}, \quad q > \frac{1}{n+1} + \frac{1}{p}(n+2)/(n+1)\Big\}.$$

Let $\mathcal{C}^-$ be the bounded subdomain of $\mathcal{U}$,

$$\mathcal{C}^- = \Big\{(p,q) \in \mathcal{U}, \quad q < \frac{1}{n+1} + \frac{1}{p}(n+2)/(n+1)\Big\};$$

let $\mathcal{C}^0$ be the curve

$$\mathcal{C}^0 = \Big\{(p,q) \in \mathcal{U}, \quad q = \frac{1}{n+1} + \frac{1}{p}(n+2)/(n+1)\Big\}.$$

Thus $\mathcal{U} = \mathcal{C}^+ \cup \mathcal{C}^0 \cup \mathcal{C}^-$.

Let $\mathcal{L}$ be the set of functions f defined on $\mathbb{R}^n \times \mathbb{R}^+$, with continuous derivatives up to order 1 in time and 2 in space, satisfying:

(i) $f(x,t) \geq 0, \ \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$.
(ii) For all $t \in \mathbb{R}^+, \ x \mapsto f(x,t) \in C_b^2(\mathbb{R}^n)$.
(iii) $\nabla_x f \in (L^\infty(\mathbb{R}^n \times \mathbb{R}^+))^n$.

When $(u,v) \in \mathcal{L} \times \mathcal{L}$ is a solution to (1), we say that u and v are classical nonnegative global solutions with bounded gradients.

Our principal theorem is the following.

THEOREM 1: *Suppose $\mu, \nu < 0$, and $1 \leq p \leq q$. Let $(u,v) \in \mathcal{L} \times \mathcal{L}$ be a solution to (1) with initial data u_0 in $L^1(\mathbb{R}^n, (1+|x|^2)^{\frac{1}{2}}dx)$ and v_0 in $L^1(\mathbb{R}^n)$. Assume that $\|u_0 + v_0\|_{L^1(\mathbb{R}^n)} > 0$.*

(i) *If $(p,q) \in \mathcal{C}^+$, then $\lim_{t\to+\infty} \|u(t) + v(t)\|_{L^1(\mathbb{R}^n)} > 0$.*
(ii) *If $(p,q) \in \mathcal{C}^-$, then $\lim_{t\to+\infty} \|u(t) + v(t)\|_{L^1(\mathbb{R}^n)} = 0$.*

Remark 1: The existence of solutions $(u,v) \in \mathcal{L} \times \mathcal{L}$ to the system (1) is studied in [AR]. The result is stated here, though it is not final; it is expected to be improved upon.

Let $p, q \geq 2$ and $u_0, v_0 \in C_b^2(\mathbb{R}^n)$ with $\lim_{|x|\to+\infty} u_0(x) = \lim_{|x|\to+\infty} v_0(x) = 0$. Set $R = \sup_{x\in\mathbb{R}^n}(u_0^2(x) + v_0^2(x) + |\nabla u_0(x)|^2 + |\nabla v_0(x)|^2)^{\frac{1}{2}}$.

(i) If $\sup(p|\nu|(2R)^{p-1}, q|\mu|(2R)^{q-1}) \leq 1$, then there is a unique classical solution (u,v) to (1). Moreover, this solution is global with bounded gradients.

(ii) Suppose in addition that $(u_0(x), v_0(x))$ lies for all $x \in \mathbb{R}^n$ in some closed ball B of $\mathbb{R}^2$ containing the origin. If its radius ρ satisfies

$$2\rho \sup(|\nu|(2R)^{p-2}, |\mu|(2R)^{q-2}) < 1,$$

then $(u(x,t), v(x,t)) \in B$, $\forall x \in \mathbb{R}^n$, $\forall t > 0$.

Note that part (ii) leads easily to the construction of nonnegative solutions.

The proof of part (i) of Theorem 1 necessitates the division of $\mathcal{C}^+$ into the two parts

$$\mathcal{C}_1^+ = \{(p,q) \in \mathcal{C}^+, p \le (n+2)/(n+1)\}$$

and

$$\mathcal{C}_2^+ = \{(p,q) \in \mathcal{C}^+, p > (n+2)/(n+1)\}.$$

Similarly, the proof of part (ii) requires the division of $\mathcal{C}^-$ into

$$\mathcal{C}_1^- = \{(p,q) \in \mathcal{C}^-, q \le (n+2)/(n+1)\}$$

and

$$\mathcal{C}_2^- = \{(p,q) \in \mathcal{C}^-, q > (n+2)/(n+1)\}$$

(cf. Fig. 1).

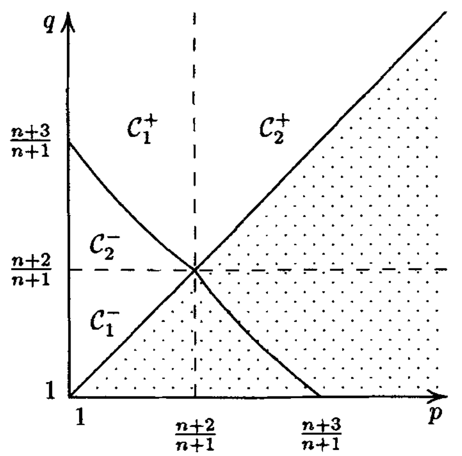

Figure 1.

More precisely we have

$$\lim_{t\to+\infty} \|u(t)\|_{L^1(\mathbb{R}^n)} = 0 \quad \text{for } 1 \le p \le \frac{n+2}{n+1}, \quad q \ge 1, \tag{4}$$

$$\lim_{t\to+\infty} \|v(t)\|_{L^1(\mathbb{R}^n)} = 0 \quad \text{for } p \ge 1, \quad 1 \le q \le \frac{n+2}{n+1}, \tag{5}$$

$$\lim_{t\to+\infty} \|u(t)+v(t)\|_{L^1(\mathbb{R}^n)} > 0 \quad \text{for } p > \frac{n+2}{n+1}, \quad q > \frac{n+2}{n+1}, \tag{6}$$

$$\lim_{t\to+\infty}\|v(t)\|_{L^1(\mathbb{R}^n)}>0 \quad \text{for } (p,q)\in\mathcal{C}_1^+, \tag{7}$$

$$\lim_{t\to+\infty}\|v(t)\|_{L^1(\mathbb{R}^n)}=0 \quad \text{for } (p,q)\in\mathcal{C}_2^-. \tag{8}$$

In particular, it follows from (4) and (7) that disymmetrical decay for u and v occurs when $(p,q)\in\mathcal{C}_1^+$. In this case, $\|u(t)\|_{L^1(\mathbb{R}^n)}$ converges to 0 while $\|v(t)\|_{L^1(\mathbb{R}^n)}$ admits a positive limit as t tends to infinity.

Note that the case $(p,q)\in\mathcal{C}^0$, with $p\neq q$, is still an open problem.

The ideas introduced by Ben-Artzi and Koch allow us to obtain the corresponding decay results for system (1), when p and q are both greater or both less than $(n+2)/(n+1)$. The other cases require a more subtle adaptation, using some disymmetric estimates.

The proof of (4) (resp. (5)) is exactly the same as the one of [BK, theorem 1.1], and we omit it. It uses the integrability at infinity of $\|\nabla u(t)\|_p^p$ (resp. $\|\nabla u(t)\|_q^q$) given by (3). This solves in particular the case $(p,q)\in\mathcal{C}_1^-$.

To prove (7) and (8) we need some precise estimate of the decay rate of $\|u(t)\|_{L^1(\mathbb{R}^n)}$ when $p<(n+2)/(n+1)$, which is given by Lemma 2. This allows us to reach the cases $(p,q)\in\mathcal{C}_1^+$ and $(p,q)\in\mathcal{C}_2^-$, using disymmetric estimates of the decay rates of the gradients of u and v.

The analysis is similar but simpler to prove (6), which leads to the case $(p,q)\in\mathcal{C}_2^+$. It is based on symmetrical decay rates for the gradients of u and v, which are the same as those in the scalar case (2). We will only sketch it.

We will now fix some notations. We will omit the notation of the x variable when there is no ambiguity. We denote by $\|\ \|_r$ the usual norm on $L^r(\mathbb{R}^n)$ for any $r\geq 1$ and $\|\nabla u(t)\|_r$ will stand for $\||\nabla_x u(\cdot,t)|\|_r$.

Here and in the sequel we use C to denote a generic constant independent of x,t. That is, C may still depend on general data of the problem $(n,p,q,u_0,v_0,\dots)$, and sometimes on some small positive parameters which appear in the statements or the proofs of certain lemmas. To emphasize certain dependencies, we shall sometimes write $C=C(\|u_0\|,\|v_0\|,\dots)$ or $C(\varepsilon)$.

For similar reasons as [BK], Duhamel's principle leads to the following inequalities which shall be used through Sections 2, 3 and 4. For $1\leq r<+\infty$

$$\|\nabla u(t)\|_r\leq Ct^{-1/2}\|u_0\|_r+C\int_0^t(t-s)^{-1/2}\||\nabla v(s)|^q\|_r\,ds, \tag{9}$$

$$\|\nabla v(t)\|_r\leq Ct^{-1/2}\|v_0\|_r+C\int_0^t(t-s)^{-1/2}\||\nabla u(s)|^p\|_r\,ds, \tag{10}$$

and

$$(11)\quad \|\nabla u(t+s)\|_\infty \le Cs^{-1/2}\|u(t)\|_\infty + C\int_0^s (s-\sigma)^{-1/2}\||\nabla v(t+\sigma)|^q\|_\infty\, d\sigma,$$

$$(12)\quad \|\nabla v(t+s)\|_\infty \le Cs^{-1/2}\|v(t)\|_\infty + C\int_0^s (s-\sigma)^{-1/2}\||\nabla u(t+\sigma)|^p\|_\infty\, d\sigma,$$

for $t > 0$, $s > 0$.

Note that the first term in the right hand side of (11) [resp. (12)] can be replaced by $Cs^{-(n+1)/2}\|u(t)\|_1$ [resp. $Cs^{-(n+1)/2}\|v(t)\|_1$].

The inequality (11) with the first term replaced by $Cs^{-(n+1)/2}\|v(t)\|_1$ and the inequality (12) are the starting point for obtaining nonsymmetric estimates for the gradients $\nabla u(t), \nabla v(t)$ in $\mathcal{C}_1^+$. In $\mathcal{C}_2^+$ we start with (11) and (12) unchanged, which gives rise to symmetric estimates for the gradients.

We shall denote by $G(x,t)$ the heat kernel, $G(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Recall that

$$(13)\quad \begin{gathered} \int_{\mathbb{R}^n} G(x,t)\, dx = 1, \quad \sup_{x\in\mathbb{R}^n} G(x,t) = (4\pi t)^{-n/2}, \\ \int_{\mathbb{R}^n} |\nabla G(x,t)|\, dx = c_n t^{-1/2}, \quad \sup_{x\in\mathbb{R}^n} |\nabla G(x,t)| = d_n t^{-(n+1)/2}, \end{gathered}$$

where

$$c_n = \Gamma\Big(\frac{n+1}{2}\Big)/\Gamma\Big(\frac{n}{2}\Big) \quad \text{and} \quad d_n = 2^{-(n+1)}\pi^{-n/2}.$$

The paper is organized as follows. Section 2 is concerned with the polynomial estimate of $\|u(t)\|_{L^1(\mathbb{R}^n)}$ to be used in Sections 3 and 5. In Section 3 we prove some estimates of the gradients of u and v to be used in Section 4. Section 4 is devoted to the case $(p,q) \in \mathcal{C}^+$ and we prove (6) and (7). Finally, in Section 5, (p,q) belongs to $\mathcal{C}_2^-$ and (8) is derived.

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2. Polynomial rate of decay of $\|u\|_{L^1}$ when $p < (n+2)/(n+1)$

From now on, u and v are classical nonnegative solutions to (1) with bounded gradients. We suppose $\mu, \nu < 0$. As already mentioned, it is possible by a proper scaling to suppose $\mu = \nu = -1$. In this section we derive a polynomial estimation of $\|u(t)\|_1$, to be used in Sections 3 and 5, when $p < (n+2)/(n+1), q \ge 1$. Let us recall that $\|u(t)\|_1$ decays to zero also for $p = (n+2)/(n+1)$ (although with no estimate of the decay).

LEMMA 1: *Let $p, q \geq 1$, and suppose that the initial data v_0 satisfies $\int_{\mathbb{R}^n} v_0(x)\,dx < \infty$. There exists $N > 0$ such that for all $t \geq 1$, there exists $\tau \in [t, Nt]$ satisfying $\|\nabla u(\cdot,\tau)\|_p \leq \tau^{-1/p}$.*

Proof: Suppose the contrary. Then

$$\forall N > 0,\ \exists t_N \geq 1,\ \forall \tau \in [t_N, Nt_N],\ \|\nabla u(\cdot,\tau)\|_p^p > \tau^{-1}. \tag{14}$$

Besides this, integrating (1) gives $\int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x,t)|^p\,dxdt < \infty$. Thus (14) shows that $\forall N > 0$,

$$\infty > \int_{t_N}^{Nt_N} \|\nabla u(\cdot,\tau)\|_p^p\,d\tau > \ln N.$$

This is a contradiction. ∎

LEMMA 2: *Let $1 \leq p < (n+2)/(n+1)$, $q \geq 1$ and let u and v be nonnegative solutions to (1) with the initial data u_0, v_0 satisfying $\int_{\mathbb{R}^n} u_0(x)(1+|x|^2)^{1/2}\,dx < \infty$ and $\int_{\mathbb{R}^n} v_0(x)\,dx < \infty$. Then for any $\varepsilon > 0$ there exists $C > 0$ such that*

$$\|u(t)\|_1 \leq Ct^{\frac{n+1}{2} - \frac{n+2}{2p} + \varepsilon}. \tag{15}$$

Proof: We first use two inequalities established in [BK].

Let $r > 0$. Using a Morrey type inequality one proves

$$\|u(t)\|_1 \leq 2r \int_{|x|\leq 3r} |\nabla u(x,t)|\,dx + 2\int_{|x|>r} |u(x,t)|\,dx \tag{16}$$

and noting that $u(t) \leq G(t) * u_0$ one derives (see [BK] for more details)

$$\int_{|x|>r} u(x,t)\,dx \leq \int_{|x|>r/2} u_0(x)\,dx + \|u_0\|_1 \int_{|x|>r/(2t^{1/2})} G(x,1)\,dx. \tag{17}$$

Then, by a change of variable

$$\begin{aligned}\int_{|x|>r/(2t^{1/2})} G(x,1)\,dx &\leq \pi^{-n/2}|S^{n-1}| \int_{\rho>r/(4t^{1/2})} e^{-\rho^2}\rho^{n-1}\,d\rho \\ &\leq C\pi^{-n/2}|S^{n-1}| \int_{\rho>r/(4t^{1/2})} e^{-\rho^2/2}\,d\rho\end{aligned}$$

where $C = \sup_{\rho\geq 0} e^{-\rho^2/2}\rho^{n-1}$.

Let $\delta > 0$. We choose $r = r(t) = t^{1/2+\delta}$. Let us mention here that a choice like $r = t^{1/2}\log^\alpha(2+t)$ for some $\alpha > 0$ is not increasing fast enough for the following purpose.

Then one gets

$$\int_{|x|>r/(2t^{1/2})} G(x,1)\,dx \le 2^{1/2} C \pi^{-n/2} |S^{n-1}| \int_{\tau > t^{\delta}/2^{5/2}} e^{-\tau^2}\,d\tau$$

and therefore

$$\int_{|x|>r/(2t^{1/2})} G(x,1)\,dx \le C e^{-t^{2\delta}/32} \tag{18}$$

for some positive real number C depending only on n. In (18) we make use of the rough inequality

$$\operatorname{erf}(a) = \int_{\rho > a} e^{-\rho^2}\,d\rho \le \frac{\pi^{1/2}}{2} e^{-a^2}, \quad \forall\, a > 0.$$

This inequality can be easily improved, but without changing the estimate of the lemma.

Next, the first term on the r.h.s. of (17) is obviously estimated by

$$\int_{|x|>r/2} u_0(x)\,dx \le 2r^{-1} \int_{\mathbb{R}^n} |x| u_0(x)\,dx. \tag{19}$$

It remains to give a bound of $2r \int_{|x|\le 3r} |\nabla u(x,t)|\,dx$. However, this is possible only for some sequence $(t_j)_{j\ge 1}$ of time. This restriction shall be eliminated later using the fact that $\|u(\cdot,t)\|_1$ is nonincreasing together with a suitable control of the sequence $(t_j)_{j\ge 1}$.

Following Lemma 1, there exists $N > 0$ and an increasing sequence $(t_j)_{j\ge1}$ with

$$\lim_{j\to+\infty} t_j = +\infty,$$

$$t_{j+1} \le N t_j, \quad \forall\, j \ge 1, \tag{20}$$

$$\|\nabla u(\cdot, t_j)\|_p \le t_j^{-1/p}, \quad \forall j \le 1. \tag{21}$$

As in [BK] it is then clear that

$$2r \int_{|x|\le 3r} |\nabla u(x,t_j)|\,dx \le 2.6^{n(1-1/p)} r^{1+n(1-1/p)} \|\nabla u(\cdot,t_j)\|_p,$$

which gives with (21)

$$2r \int_{|x|\le 3r} |\nabla u(x,t_j)|\,dx \le d_n t_j^{\frac{n+1}{2} - \frac{n+2}{2p} + \delta(n+1-\frac{n}{p})}. \tag{22}$$

Combining (16), (17) with (18), (19) for $t = t_j$ and (22) gives

$$\|u(\cdot,t_j)\|_1 \le d_n t_j^{\frac{n+1}{2}-\frac{n+2}{2p}+\delta(n+1-\frac{n}{p})} + c_n e^{-t_j^{2\delta}/32} + \|xu_0(x)\|_1 t_j^{-\frac{1}{2}-\delta} \tag{23}$$

Since $p \ge 1$, we have $-\frac{1}{2} \le (n+1)/2 - (n+2)/2p$, in such a way that

$$t_j^{-\frac{1}{2}-\delta} = O(t_j^{\frac{n+1}{2}-\frac{n+2}{2p}+\delta(n+1-\frac{n}{p})}).$$

Moreover,

$$e^{-t_j^{2\delta}/32} = O(t_j^{\frac{n+1}{2}-\frac{n+2}{2p}+\delta(n+1-\frac{n}{p})}).$$

Thus, for any $\varepsilon > 0$, we deduce that

$$\|u(\cdot,t_j)\|_1 \le C t_j^{\frac{n+1}{2}-\frac{n+2}{2p}+\varepsilon} \tag{24}$$

for large j.

Any positive number t lies in one of the intervals $[t_j, t_{j+1})$. Since $\|u(t)\|_1$ is non-increasing, (20) and (21) finally give

$$\|u(t)\|_1 \le C(t/N)^{\frac{n+1}{2}-\frac{n+2}{2p}+\varepsilon},$$

where C does not depend on j. This proves Lemma 2. ∎

We always suppose in the following that $u_0 \in L^1(\mathbb{R}^n, (1+|x|^2)^{\frac{1}{2}}dx)$ and $v_0 \in L^1(\mathbb{R}^n)$.

Remark 2: In the previous proof, one sees that (16), (17), (18) (with $r(t) = t^{1/2+\delta}$) does not depend on the assumption $p < (n+2)/(n+1)$. Moreover, $\int_{|x|>R} u_0(x)\,dx$ tends obviously to 0 as $R \to +\infty$. In this way we have $\int_{|x|>r(t)} u(x,t)\,dx \to 0$ as $t \to +\infty$ (besides, this argument appears in Lemma 2.1 of [BK]). As a consequence, we will be able to use the analogous property for v, in Section 5.

3. L^∞ and L^1 estimates of the gradients in $\mathcal{C}^+$

This section is devoted to a careful inspection of the behaviour in time of the L^1 and L^∞ norms for ∇u and ∇v.

Lemmas 4 and 5 are concerned with the case $(p,q) \in \mathcal{C}_1^+$, and Lemma 6 with the case $(p,q) \in \mathcal{C}_2^+$. All of them are used in the next section.

We start with the case $(p,q) \in \mathcal{C}_1^+$. First we look at the L^∞ norm of the gradients. The estimate derived in Lemma 2 will serve us in the proof of the following lemma.

LEMMA 3: *Let* $q \geq 1$.

1. *Let* $p \in [1, (n+2)/(n+1)]$. *If* $\|\nabla u(t)\|_\infty \leq Ct^{-\alpha}$ *with* $\alpha \in [0, \frac{n+2}{2p}]$, *then* $\|\nabla v(t)\|_\infty \leq Ct^{-\beta}$ *with* $\beta = \frac{n}{4} + \frac{p}{2}\alpha$.

2. *Let* $p \in [1, (n+2)/(n+1))$. *If* $\|\nabla v(t)\|_\infty \leq Ct^{-\beta}$ *with* $\beta \in [0, \frac{n+2+p}{2pq})$, *then for any* $\varepsilon > 0$, $\|\nabla u(t)\|_\infty \leq Ct^{-\alpha}$ *with* $\alpha = \frac{1}{2p} - \frac{1}{2}\frac{n+1}{n+2} + q\frac{n+1}{n+2}\beta - \varepsilon$.

3. *Let* $p = (n+2)/(n+1)$. *If* $\|\nabla v(t)\|_\infty \leq Ct^{-\beta}$ *with* $\beta \in [0, \frac{n+2+p}{2pq})$, *then* $\|\nabla u(t)\|_\infty \leq Ct^{-\alpha}$ *with* $\alpha = \frac{1}{2p} - \frac{1}{2}\frac{n+1}{n+2} + q\frac{n+1}{n+2}\beta$.

Remark 3: In part 3 it is convenient for the sequel not to replace p by $(n+2)/(n+1)$ in α.

Proof: 1. In view of (12), we have

$$\|\nabla v(t)\|_\infty \leq Cs^{-\frac{1}{2}}\|v(\tau)\|_\infty + C\int_0^s (s-\sigma)^{-1/2}\|\nabla u(\tau+\sigma)\|_\infty^p \, d\sigma,$$

for $s > 0$, $\tau > 0$ with $s + \tau = t$. Since $v(\cdot, t)$ is bounded by $G(\cdot, t) * v_0$, using the given assumption on $\|\nabla u(t)\|_\infty$ we have

$$\|\nabla v(t)\|_\infty \leq Cs^{-\frac{1}{2}}\tau^{-n/2} + Cs^{\frac{1}{2}}\tau^{-p\alpha}. \tag{25}$$

We now look for s and t such that the two terms in the r.h.s. of (25) are equal. This occurs when

$$\tau + \tau^\gamma = t \tag{26}$$

where $\gamma = -n/2 + p\alpha$. By hypothesis $\gamma \in [-n/2, 1]$.

Case a: $\gamma \in [0, 1]$. There exists a unique $\tau = \tau(t)$ satisfying (26). This τ verifies $t/2 \leq \tau \leq t$ for t large enough ($t \geq 2$).

Case b: $\gamma \in [-n/2, 0)$. There exist two solutions $\tau_1(t) < \tau_2(t)$ to (26), satisfying $\tau_1(t)^\gamma \sim t$ and $\tau_2(t) \sim t$ near infinity. We then keep $\tau = \tau_2(t)$.

In both cases, for t sufficiently large, there exists a solution τ of (26) satisfying $\tau \in [t/2, t]$. With this τ

$$s^{-\frac{1}{2}}\tau^{-n/2} = s^{\frac{1}{2}}\tau^{-p\alpha} = \tau^{-(\frac{n}{4} + \frac{p\alpha}{2})}.$$

Then

$$\|\nabla v(t)\|_\infty \leq Ct^{-(\frac{n}{4} + \frac{p\alpha}{2})}.$$

2. The proof is similar. Here $p \in [1, (n+2)/(n+1))$. Following (11) along with the comments, and using the assumption on $\|\nabla v(t)\|_\infty$ and the estimate (15) of Lemma 2, we deduce

$$\|\nabla u(t)\|_\infty \leq Cs^{-\frac{n+1}{2}}\tau^{\frac{n+1}{2} - \frac{n+2}{2p} + \varepsilon} + Cs^{\frac{1}{2}}\tau^{-q\beta} \tag{27}$$

where $s, \tau > 0$, $s + \tau = t$ and for any $\varepsilon > 0$. The two terms in the r.h.s. of (27) are equal when $\tau + \tau^{\gamma_\varepsilon} = t$ where

$$\gamma_\varepsilon = \gamma + \frac{2\varepsilon}{n+2} = \frac{n+1}{n+2} - \frac{1}{p} + \frac{2q}{n+2}\beta + \frac{2\varepsilon}{n+2}.$$

The hypothesis $\beta < (n+2+p)/2pq$ gives $\gamma \in (-\infty, 1)$. Now fix ε with $\gamma_\varepsilon = \gamma + 2\varepsilon/(n+2) < 1$. Repeating the same arguments as those in part 1 we obtain the existence of $\tau = \tau(t, \varepsilon)$ satisfying $\tau \in [t/2, t]$ and $\tau + \tau^{\gamma_\varepsilon} = t$ for large t. Then

$$\|\nabla u(t)\|_\infty \leq Ct^{-\frac{1}{2p} + \frac{1}{2}\frac{n+1}{n+2} - q(n+2)/(n+1)\beta + \frac{\varepsilon}{n+2}}. \tag{28}$$

This gives the stated estimate for any $0 < \varepsilon < 1 - \gamma$ and then for every $\varepsilon > 0$.

3. Here $p = (n+2)/(n+1)$. We use the fact that $\|u(t)\|_1$ is bounded instead of Lemma 2. Then (27) holds replacing ε by 0 and (28) is true with $\varepsilon = 0$. ∎

We proceed now to establish a $L^\infty(\mathbb{R}^n)$ bound for $\nabla u(t)$ and $\nabla v(t)$.

LEMMA 4: *Let $(p, q) \in \mathcal{C}_1^+$. For any $\varepsilon > 0$ we have*

$$\|\nabla u(t)\|_\infty \leq Ct^{-\frac{n+2}{2p} + \varepsilon} \quad \text{and} \quad \|\nabla v(t)\|_\infty \leq Ct^{-\frac{n+1}{2} + \varepsilon}.$$

Proof: Let us recall that (u, v) is supposed to be in $\mathcal{L} \times \mathcal{L}$, and therefore ∇u is uniformly bounded. The proof is a consequence of Lemma 3. Here p, q, n are fixed with $(p, q) \in \mathcal{C}_1^+$.

Case I: $p \in [1, (n+2)/(n+1))$ *and* $pq < 2\frac{n+2}{n+1}$. Observe that we have

$$\frac{2\frac{n+2}{p(n+1)} - 2 + qn}{2(2(n+2)/(n+1) - pq)} - \frac{n+2}{2p} > 0 \Leftrightarrow (p, q) \in \mathcal{C}^+,$$
$$\frac{n+2-p}{2(2(n+2)/(n+1) - pq)} - \frac{n+1}{2} > 0 \Leftrightarrow (p, q) \in \mathcal{C}^+.$$

Then fix $\varepsilon > 0$ satisfying

$$\frac{2\frac{n+2}{p(n+1)} - 2 + qn}{2(2\frac{n+2}{n+1} - pq)} - \frac{n+2}{2p} > \frac{4\varepsilon\frac{n+2}{n+1}}{2(2\frac{n+2}{n+1} - pq)}, \tag{29}$$

$$\frac{n+2-p}{2(2\frac{n+2}{n+1} - pq)} - \frac{n+1}{2} > \frac{2\frac{n+2}{n+1}p\varepsilon}{2(2\frac{n+2}{n+1} - pq)}, \tag{30}$$

$$\frac{n+2+p}{2pq} \geq \varepsilon, \tag{31}$$

$$\frac{n+2}{2p} \geq \left(1 + q\frac{n+1}{n+2}\right)\varepsilon. \tag{32}$$

With this ε and in view of Lemma 3 we can start a bootstrap with $\|\nabla u(t)\|_\infty$ and $\|\nabla v(t)\|_\infty$. To do this we consider the sequence $(\alpha_j)_{j\geq 0}$ of real numbers defined by

$$\begin{aligned}\alpha_{2j} &= \frac{1}{2p} - \frac{n+1}{2(n+2)} + q\frac{n+1}{n+2}\alpha_{2j-1} - \varepsilon,\\ \alpha_{2j+1} &= \frac{n}{4} + \frac{p}{2}\alpha_{2j},\end{aligned} \tag{33}$$

with $\alpha_0 = 0$.

The subsequences $(\alpha_{2j})_{j\geq 0}$ and $(\alpha_{2j+1})_{j\geq 0}$ may also be defined as follows:

$$\begin{cases}\alpha_{2j} = \frac{1}{2p} - \frac{n+1}{2(n+2)} + \frac{qn(n+1)}{4(n+2)} + \frac{pq(n+1)}{2(n+2)}\alpha_{2j-2} - \varepsilon,\\ \alpha_0 = 0,\end{cases} \tag{34}$$

and

$$\begin{cases}\alpha_{2j+1} = \frac{n+1}{4} - \frac{p(n+1)}{4(n+2)} + \frac{pq(n+1)}{2(n+2)}\alpha_{2j-1} - \frac{p\varepsilon}{2},\\ \alpha_1 = \frac{n}{4}.\end{cases} \tag{35}$$

Since $(p,q) \in \mathcal{C}_1^+$, we have $p \leq (n+2)/(n+1) < q$, in such a way that

$$\alpha_2 = \frac{1}{2p} - \frac{n+1}{2(n+2)} + \frac{qn(n+1)}{4(n+2)} - \varepsilon > 0 = \alpha_0, \tag{36}$$

$$\alpha_3 = \frac{n+1}{4} - \frac{p(n+1)}{4(n+2)} + \frac{pq(n+1)}{2(n+2)}\frac{n}{4} - \frac{p\varepsilon}{2} > \frac{n}{4} = \alpha_1, \tag{37}$$

provided that ε satisfies

$$\varepsilon < \frac{n}{4}. \tag{38}$$

We then deduce from (36) and (37) together with the fact that $pq(n+1)/2(n+2) > 0$ that the subsequences $(\alpha_{2j})_{j\geq 0}$ and $(\alpha_{2j+1})_{j\geq 0}$ are increasing.

From $pq(n+1)/2(n+2) < 1$ we deduce that $\lim_{j\to+\infty}\alpha_{2j} = l_1$ and $\lim_{j\to+\infty}\alpha_{2j+1} = l_2$, with

$$l_1 = \frac{2\frac{n+2}{p(n+1)} - 2 + qn - 4(n+2)/(n+1)\varepsilon}{2\left(2(n+2)/(n+1) - pq\right)}$$

and

$$l_2 = \frac{n+2-p-2(n+2)/(n+1)p\varepsilon}{2\left(2(n+2)/(n+1) - pq\right)}.$$

Moreover, (29) and (30) imply

$$l_1 > \frac{n+2}{2p}, \tag{39}$$

$$l_2 > \frac{n+1}{2}. \tag{40}$$

In (39), we write strict inequalities since we are allowed only a finite number of iterations in the bootstrap.

In view of (39) and (40), we know that $(\alpha_{2j}, \alpha_{2j+1})$ does not remain in $[0, \frac{n+2}{2p}] \times [0, \frac{n+2+p}{2pq})$ for all $j \geq 0$.

Suppose that the first out is α_{2j_0}: that is to say,

$$(\alpha_{2j}, \alpha_{2j+1}) \in [0, \frac{n+2}{2p}] \times [0, \frac{n+2+p}{2pq}), \forall j < j_0 \quad \text{and} \quad \alpha_{2j_0} > \frac{n+2}{2p}.$$

Then take $\alpha = (n+2)/2p$ in Lemma 3(1) to obtain $\beta = (n+1)/2$. Consequently, Lemma 4 is valid with $\varepsilon = 0$. Actually this case never occurs, however it is easier to consider it rather than to show that it never happens.

The remaining case is when α_{2j_1+1} is the first out. It is in this case that we lose an $\varepsilon > 0$ in the estimates. Indeed

$$(\alpha_{2j-1}, \alpha_{2j}) \in [0, \frac{n+2+p}{2pq}) \times [0, \frac{n+2}{2p}], \forall j \leq j_1 \quad \text{and} \quad \alpha_{2j_1+1} \geq \frac{n+2+p}{2pq}.$$

Then take

$$\beta = \frac{n+2+p}{2pq} - \varepsilon \tag{41}$$

in Lemma 3(2) since $\beta \geq 0$ by (31). We obtain

$$\begin{aligned} \alpha =& \frac{1}{2p} - \frac{1}{2}\frac{n+1}{n+2} + q\frac{n+1}{n+2}\beta - \varepsilon \\ =& \frac{n+2}{2p} - \Big(1 + q\frac{n+1}{n+2}\Big)\varepsilon. \end{aligned} \tag{42}$$

By (32), we have $\alpha \geq 0$. Then apply Lemma 3(1) to obtain a new exponent

$$\beta = \frac{n}{4} + \frac{p}{2}\alpha = \frac{n+1}{2} - \frac{p}{2}\Big(1 + q\frac{n+1}{n+2}\Big)\varepsilon. \tag{43}$$

Case II: $p \in [1, (n+2)/(n+1))$ *and* $pq \geq 2\frac{n+2}{n+1}$. It is a straightforward modification of the previous proof. Here we have $\lim_{j\to+\infty} \alpha_{2j} = \lim_{j\to+\infty} \alpha_{2j+1} = +\infty$. The required assumptions on ε are only (38), (31) and (32).

Case III: $p = (n+2)/(n+1)$. It is similar to Cases I and II, but simpler. Indeed, invoking Lemma 3(3) instead of Lemma 3(2), $(\alpha_j)_{j\geq 0}$ is defined as in (33) replacing ε by 0. No ε appears until (41). The ε in (41) needs to satisfy only (31) and (32). Then (42) and (43) are valid for $p = (n+2)/(n+1)$.

In all cases, inequalities (42) and (43) prove Lemma 4. ∎

Next we look at the L^1 bound of the gradients, in terms of $\|u(t)\|_1$ and $\|v(t)\|_1$.

LEMMA 5: *Let $(p,q) \in \mathcal{C}_1^+$. Then*

$$\|\nabla u(t+s)\|_1 \leq C\left(s^{-\frac{1}{2}}\|u(t)\|_1 + t^{-\frac{n+1}{2}(q-1)+\varepsilon}\|v(t)\|_1\right) \tag{44}$$

and

$$\|\nabla v(t+s)\|_1 \leq C\left(t^{-\frac{n+2}{2p}(p-1)+\varepsilon}\|u(t)\|_1 + +s^{-\frac{1}{2}}\|v(t)\|_1\right) \tag{45}$$

hold for large t and $0 < s \leq t$.

Proof: Observe that we have, for $p \leq q$,

$$\frac{n+2}{2p}(p-1) + \frac{n+1}{2}(q-1) > 1 \Leftrightarrow (p,q) \in \mathcal{C}^+.$$

Then fix $\varepsilon > 0$ satisfying

$$1 + 2\varepsilon < \frac{n+2}{2p}(p-1) + \frac{n+1}{2}(q-1). \tag{46}$$

Following (9)(10) and Lemma 4 we have for any $t, s > 0$

$$\|\nabla u(t+s)\|_1 \leq Cs^{-\frac{1}{2}}\|u(t)\|_1 + Ct^{-\frac{n+1}{2}(q-1)+\varepsilon}\int_0^s (s-\sigma)^{-\frac{1}{2}}\|\nabla v(t+\sigma)\|_1\, d\sigma \tag{47}$$

and

$$\|\nabla v(t+s)\|_1 \leq Cs^{-\frac{1}{2}}\|v(t)\|_1 + Ct^{-\frac{n+2}{2p}(p-1)+\varepsilon}\int_0^s (s-\sigma)^{-\frac{1}{2}}\|\nabla u(t+\sigma)\|_1\, d\sigma. \tag{48}$$

Define

$$g(s) = \sup_{0\leq\sigma\leq s} \sigma^{1/2}\|\nabla u(t+\sigma)\|_1,\ h(s) = \sup_{0\leq\sigma\leq s} \sigma^{1/2}\|\nabla v(t+\sigma)\|_1.$$

Plugging (48) into (47) we derive an inequality involving $g(s)$ only:

$$\begin{aligned} g(s) \leq & C\|u(t)\|_1 + Cs^{\frac{1}{2}}t^{-\frac{n+1}{2}(q-1)+\varepsilon}\|v(t)\|_1 \\ & + Cst^{-\frac{n+2}{2p}(p-1)-\frac{n+1}{2}(q-1)+2\varepsilon}g(s). \end{aligned} \tag{49}$$

Similarly, the following inequality involving $h(s)$ holds:

$$
\begin{aligned}
(50) \qquad h(s) \leq & C\|v(t)\|_1 + Cs^{\frac{1}{2}}t^{-\frac{n+2}{2p}(p-1)+\varepsilon}\|u(t)\|_1 \\
& + Cst^{-\frac{n+2}{2p}(p-1)-\frac{n+1}{2}(q-1)+2\varepsilon}h(s).
\end{aligned}
$$

From (46), we obtain for large t and $s \leq t$

$$
Cst^{-\frac{n+2}{2p}(p-1)-\frac{n+1}{2}(q-1)+2\varepsilon} \leq \frac{1}{2}.
$$

Therefore (49)–(50) prove Lemma 5. ∎

Remark 4: It is easy to see that (44) and (45) still hold for $t \geq t_1$ and $s \leq \eta t$, where t_1 is some fixed positive real number, and η depends only on t_1. This property will be used below, in the proof of Lemma 9.

The case $(p,q) \in \mathcal{C}_2^+$ remains to be dealt with. The analysis is analogous but simpler than the previous one. Indeed $\nabla u(t)$ and $\nabla v(t)$ are subject to the same treatment and no ε appears. As a matter of fact, the decay rates of both $\nabla u(t)$ and $\nabla v(t)$ in the L^∞ norm are the same as those in the scalar case (2) and therefore do not depend on the parameters p and q. We obtain the analogous results as those of Lemmas 4 and 5. They are stated in the following lemma, where the assumption $p \leq q$ is shown to be needless.

LEMMA 6: *Suppose $p, q > (n+2)/(n+1)$. Then*
(i) $\|\nabla u(t)\|_\infty \leq Ct^{-\frac{n+1}{2}}$, $\|\nabla v(t)\|_\infty \leq Ct^{-\frac{n+1}{2}}$.
For large t and $s \leq t$, we have
(ii) $\|\nabla u(t+s)\|_1 \leq C\left(s^{-\frac{1}{2}}\|u(t)\|_1 + t^{-\frac{n+1}{2}(q-1)}\|v(t)\|_1\right)$,
(iii) $\|\nabla v(t+s)\|_1 \leq C\left(t^{-\frac{n+1}{2}(p-1)}\|u(t)\|_1 + s^{-\frac{1}{2}}\|v(t)\|_1\right)$.

4. The case $(p,q) \in \mathcal{C}^+$

In this section we establish (6) and (7), that is to say, the positivity of the limit of $\|u(t)+v(t)\|_1$ for $(p,q) \in \mathcal{C}_1^+$ or $(p,q) \in \mathcal{C}_2^+$, if $\|u_0+v_0\|_1 > 0$.

We start with the case $(p,q) \in \mathcal{C}_1^+$: the proof of (7) results from Lemmas 7 to 9.

Let us recall here that $\|u(t)\|_1$ tends to 0 at infinity in view of (4) and that $\|v(t)\|_1$ admits a nonnegative limit.

LEMMA 7: *Let* $(p,q) \in \mathcal{C}_1^+$.

If $\|v(t)\|_1 \to 0$ *as* $t \to +\infty$, *then* $\|u(t)\|_1$, $\|v(t)\|_1$, $\|\nabla u(t)\|_\infty$ *and* $\|\nabla v(t)\|_\infty$ *decay faster than any negative power of* t.

Proof: Let $I(t) = \|u(t)\|_1$ and $J(t) = \|v(t)\|_1$. Integrating (1) gives, for all $t > 0$,

$$I_t(t) \geq -C\|\nabla v(t)\|_\infty^{q-1} \|\nabla v(t)\|_1, \tag{51}$$

$$J_t(t) \geq -C\|\nabla u(t)\|_\infty^{p-1} \|\nabla u(t)\|_1. \tag{52}$$

Thus, from Lemma 4 and Lemma 5, we get for large t

$$-I_t(t) \leq C\left(t^{-1-\delta} I(t/2) + t^{-\alpha} J(t/2)\right), \tag{53}$$

$$-J_t(t) \leq C\left(t^{-\beta} I(t/2) + t^{-1-\delta} J(t/2)\right), \tag{54}$$

where

$$\alpha = \frac{n+1}{2}(q-1) + \frac{1}{2} - \varepsilon, \quad \beta = \frac{n+2}{2p}(p-1) + \frac{1}{2} - \varepsilon,$$
$$1 + \delta = \frac{n+1}{2}(q-1) + \frac{n+2}{2p}(p-1) - 2\varepsilon.$$

Since $(p,q) \in \mathcal{C}_1^+$, we can choose ε in such a way that

$$\alpha > 1, \quad \beta < 1, \quad \delta > 0, \quad \alpha + \beta = 2 + \delta. \tag{55}$$

Then we use the boundedness of I and J, together with the inequality $1+\delta < \alpha$, to obtain from (53)

$$-I_t(t) \leq -Ct^{-1-\delta},$$

and, after integrating over $[t, +\infty)$,

$$I(t) \leq Ct^{-\delta}.$$

Iterating this process, we find

$$-I_t(t) \leq -C(t^{-1-2\delta} + t^{-\alpha}) \quad \text{and} \quad I(t) \leq Ct^{-\min(2\delta;\alpha-1)}.$$

After a finite number of steps, this leads to the estimate

$$I(t) \leq Ct^{1-\alpha}. \tag{56}$$

Plugging this estimate into (54), and noting that $1 - \alpha - \beta = -1 - \delta$, gives after integration

$$J(t) \leq Ct^{-\delta}. \tag{57}$$

We now use the same techniques to prove by iteration that

$$I(t) \leq Ct^{1-\alpha-k\delta} \quad \text{and} \quad J(t) \leq Ct^{-k\delta}. \tag{58}$$

for all integers k. This establishes the stated decay for $\|u(t)\|_1$ and $\|v(t)\|_1$.

For proving the corresponding result relative to $\|\nabla u(t)\|_\infty$ and $\|\nabla v(t)\|_\infty$, we use the same method as in the proof of Lemma 3.

Suppose that $\|u(t)\|_1 + \|v(t)\|_1 \leq Ct^{-k}$, for some integer k.

Using Lemma 4 after setting $t = s + \tau$, it is easy to obtain, as for (27),

$$\|\nabla u(t)\|_\infty \leq Cs^{-\frac{n+1}{2}}\tau^{-k} + Cs^{1/2}\tau^{-q(\frac{n+1}{2}-\varepsilon)}.$$

The two terms in the r.h.s. member are equal if τ satisfies $\tau + \tau^\gamma = t$, where

$$\gamma = \frac{(n+1-2\varepsilon)q - 2k}{n+2}.$$

This exponent is less than 1 for sufficiently large k. For all $t > 0$, we can define $\tau = \tau(t)$ satisfying $\tau + \tau^\gamma = t$, in such a way that $\tau(t) \sim t$ near infinity. We then obtain

$$\|\nabla u(t)\|_\infty \leq Ct^{-\frac{k}{n+2} - \frac{(n+1)(n+1-2\varepsilon)q}{2(n+2)}}. \tag{59}$$

A similar computation gives

$$\|\nabla v(t)\|_\infty \leq Ct^{-\frac{k}{n+2} - \frac{n+1}{2} + \frac{(n+1)p\varepsilon}{n+2}}. \tag{60}$$

Taking a sufficiently large k, we obtain the stated decay. ∎

LEMMA 8: *Let $(p,q) \in \mathcal{C}_1^+$ and suppose that $\|u(t)\|_1$, $\|v(t)\|_1$, $\|\nabla u(t)\|_\infty$ and $\|\nabla v(t)\|_\infty$ decay faster than any negative power of t. Then*

$$\|u(t_0)\|_1 = \|v(t_0)\|_1 = 0 \tag{61}$$

for some $t_0 > 0$.

Proof: Using the same argument as [BK] (Lemma 3.4), we prove the existence of some $t_0 > 0$ such that

$$\|u(2s)\|_1 + \|v(2s)\|_1 \geq \frac{1}{2}(\|u(s)\|_1 + \|v(s)\|_1) \tag{62}$$

for all $s \geq t_0$.

Let us fix $t > 0$. Defining $g(s)$ as in the proof of Lemma 5, and using the estimate

$$\|u(t)\|_1 + \|v(t)\|_1 + \|\nabla u(t)\|_\infty + \|\nabla v(t)\|_\infty \le Ct^{-k},$$

we get

$$g(s) \le C\|u(t)\|_1 + Cs^{1/2}t^{-k(q-1)}\|v(t)\|_1 + Cst^{-k(p+q-2)}g(s).$$

Choosing k sufficiently large, we obtain for large t and $0 \le s \le t$

$$\|\nabla u(t+s)\|_1 \le Cs^{-1/2}\|u(t)\|_1 + Ct^{-k(q-1)}\|v(t)\|_1.$$

Integrating (1), it follows that for large s

$$\begin{aligned} &\|v(s)\|_1 - \|v(2s)\|_1 \\ \le &\int_s^{2s} \|\nabla u(\sigma)\|_\infty^{p-1}\|\nabla u(\sigma)\|_1\, d\sigma \\ \le &C\int_s^{2s} \sigma^{-k(p-1)}\left((\sigma-s)^{-1/2}\|u(s)\|_1 + s^{-k(q-1)}\|v(s)\|_1\right) d\sigma \\ \le &Cs^{-k(p-1)+1/2}\|u(s)\|_1 + Cs^{-k(p+q-2)+1}\|v(s)\|_1. \end{aligned}$$

Therefore it is easy to obtain, choosing k sufficiently large, that

$$\|v(s)\|_1 - \|v(2s)\|_1 \le \frac{1}{4}(\|u(s)\|_1 + \|v(s)\|_1)$$

holds for large s.

Similarly,

$$\|u(s)\|_1 - \|u(2s)\|_1 \le \frac{1}{4}(\|u(s)\|_1 + \|v(s)\|_1)$$

holds for large s.

Adding the previous inequalities proves (62).

To achieve the proof, we establish that t_0 satisfies (61).

Set $K(t) = \|u(t)\|_1 + \|v(t)\|_1$, and for all integers j, $s_j = 2^j t_0$. It follows from (62) that

$$K(s_j) \ge 2^{-j}K(t_0) = s_j^{-1}t_0K(t_0).$$

Since K is assumed to decay faster than any power of t, we have

$$\lim_{s\to+\infty} s_jK(s_j) = 0,$$

and $K(t_0) = 0$. ∎

LEMMA 9: *Let $(p,q) \in \mathcal{C}_1^+$ and suppose that (61) holds for some $t_0 > 0$. Then $u = v \equiv 0$.*

Proof: As shown in Remark 4, the inequalities obtained in Lemma 5 are still valid for all t greater than some fixed t_1 (say $t_1 = t_0/2$), and $s \leq \eta t$, where η only depends on t_1 and may be supposed less than 1.

Let us fix $t \geq t_0/(1+\eta) \geq t_0/2$, and compute as in the proof of Lemma 7. For all $s \leq \eta t$, we obtain with the same notations

$$-I_t(t+s) \leq C\left(t^{-1-\delta}I(t) + s^{-1/2}t^{-\alpha+1/2}J(t)\right), \tag{63}$$

$$-J_t(t+s) \leq C\left(s^{-1/2}t^{-\beta+1/2}I(t) + t^{-1-\delta}J(t)\right). \tag{64}$$

Noticing that $\beta < 1 < \alpha$, we show that the function $K = I + J$ satisfies

$$-K_t(t+s) \leq C\left(t^{-1-\delta} + s^{-1/2}t^{-\beta+1/2}\right)K(t).$$

Since $K(t_0) = 0$ and $t_0 - t \leq \eta t$, we have

$$\begin{aligned}K(t) &= -\int_0^{t_0-t} K_t(t+s)\,ds \\ &\leq CK(t)\left(t^{-1-\delta}(t_0-t) + 2t^{-\beta+1/2}(t_0-t)^{1/2}\right) \\ &\leq C\left((t_0/2)^{-1/2-\delta} + 2(t_0/2)^{-\beta+1/2}\right)(t_0-t)^{1/2}K(t).\end{aligned}$$

For t close enough to t_0, we get

$$C\left((t_0/2)^{-1/2-\delta} + 2(t_0/2)^{-\beta+1/2}\right)(t_0-t)^{1/2} < 1,$$

thus $K(t) = 0$. So $K^{-1}(\{0\})$ is a nonempty, closed and open subset of $(0, +\infty)$. As a consequence, $K \equiv 0$. ∎

We are now able to prove the result given in (7). It is now similar to [BK].

Suppose that $\|u(t) + v(t)\|_1 \to 0$ as $t \to +\infty$, with $(p,q) \in \mathcal{C}_1^+$. By Lemma 7 the decay of $\|v(t)\|_1$ is faster than any power of t. Then Lemma 8 shows that $\|u(t_0) + v(t_0)\|_1$ is zero for some $t_0 \geq 0$. Lemma 9 implies then by continuity that $\|u_0 + v_0\|_1 = 0$. ∎

In the remaining case $(p,q) \in \mathcal{C}_2^+$ we follow the same process, to prove that the assumption $\|u(t) + v(t)\|_1 \to 0$ as $t \to +\infty$ implies $\|u_0 + v_0\|_1 = 0$. We only need to replace the results of Lemmas 4 and 5 by those of Lemma 6.

5. The case $(p,q) \in \mathcal{C}_2^-$

We prove here the validity of (8), i.e., that $\|v(t)\|_1 \to 0$ as $t \to +\infty$ if $(p,q) \in \mathcal{C}_2^-$.

Let $(p,q) \in \mathcal{C}_2^-$. Note that the conditions $pq < (n+2+p)/(n+1)$ and $q > (n+2)/(n+1)$ imply $p < (n+2)/(n+1)$, in such a way that (15) holds.

We fix some $\delta > 0$ and set $r(t) = t^{\frac{1}{2}+\delta}$.

As shown in Remark 2, we have for all $t > 0$

$$\|v(t)\|_1 \leq 2r(t) \int_{|x|\leq 3r(t)} |\nabla v(x,t)|\, dx + 2 \int_{|x|>r(t)} v(x,t)\, dx,$$

with

$$\lim_{t\to+\infty} \int_{|x|>r(t)} v(x,t)\, dx = 0.$$

Suppose that $\lim_{t\to+\infty} \|v(t)\|_1 > 0$. We then have for large t

$$0 < C \leq r(t) \int_{|x|\leq 3r(t)} |\nabla v(x,t)|\, dx,$$

which gives, by using the Hölder inequality,

$$0 < C \leq t^{(n+1-n/q)(1/2+\delta)} \|\nabla v(t)\|_q.$$

Thus, with the notations introduced in Lemma 7,

$$-I'(t) = \|\nabla v(t)\|_q^q \geq C t^{(n-(n+1)q)(1/2+\delta)}$$

and, after integrating on $(t,+\infty)$ (since $I(t) \to 0$ and $(n-(n+1)q)(\frac{1}{2}+\delta) < -1$),

$$I(t) \geq C t^{(n+2-(n+1)q)(1/2+\delta)}.$$

Combining with (15), this leads to

$$(n+2-(n+1)q)\Big(\frac{1}{2}+\delta\Big) \leq \frac{n+1}{2} - \frac{n+2}{2p} + \varepsilon,$$

that is,

$$\frac{n+2+p}{n+1} \leq pq + 2p\left(\frac{\varepsilon}{n+1} + \delta\Big(q - \frac{n+2}{n+1}\Big)\right).$$

Choosing ε and δ small enough, this contradicts the assumption $(p,q) \in \mathcal{C}_2^-$ and achieves the proof. ∎

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